Analysis of Turbulence by Shadowgraph

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Shadowgraphs of turbulent flows are often employed to determine the statistical structure of the density fluctuations. Previous solutions for the optical response have employed geometric optics and the assumption that the flow thickness is small compared to its distance to the shadowgraph plate. In this paper the electromagnetic equations for the shadowgraph response are solved without the above restrictions and a response function is determined for arbitrary shadowgraph geometry.

I. Introduction

THE shadowgraph method has been extensively employed in the analysis of the turbulent wakes of models of hypersonic vehicles. These analyses[1-3] have been based upon the work of Uberoi and Kovasznay[4] who determined the optical response equation for the autocorrelation of irradiance (intensity) fluctuations of a plane optical wave traversing a turbulent dielectric. Uberoi and Kovasznay based their method upon a formula first derived by Weyl.[5] The Weyl formula is based upon the use of geometric optics and assumes that the region of dielectric turbulence is thin compared to its distance to the shadowgraph plate. This assumption is not in accord with practice.

The assumption of a thin turbulent region was used by Weyl in the following sense: the individual rays follow straightline paths through the turbulent slab, but leave the region at angles determined by the line integral of the derivative of refractive index fluctuation through the medium. Irradiance fluctuations now occur because of the subsequent divergence of ray tubes outside the medium. The autocorrelation of the irradiance fluctuation is then easily shown to be proportional to the slab thickness times the square of the distance to the photographic plate. (It is also assumed that the distance to the plate is such that no caustics are formed. This assumption is discussed analytically in an appendix.)

The Weyl procedure obviously does not account for fluctuations of the ray tube diameters within the turbulent medium. It is known[6] in fact that within a turbulent medium the variance of irradiation fluctuations varies as the cube of the distance in the approximation of geometrical optics.

In this paper, we provide a rigorous treatment beginning with the wave equation for the case of plane wave incident upon a plane slab of turbulent dielectric of arbitrary thickness d and obtain the autocorrelation of irradiation at a distance $D \geq d$ from the front face. Using the Born approximation we obtain the result that the autocorrelation of irradiation is the sum of the autocorrelation of irradiation within the slab plus the autocorrelation due to ray tube fluctuations effected in back of the slab due to differences in angle of departure. The latter term, however, must be multiplied by a geometric factor, $(D - d)/D$, which accounts for slab thickness.

II. Analysis

We consider a plane monochromatic wave $u_0 = \exp\{i(k_0\bar{\epsilon}z - \omega t)\}$ incident from $z = -\infty$ with vacuum wave number $k_0 = \omega/c$ in a medium with dielectric constant $\bar{\epsilon}$. In the infinite slab between $0 \leqslant z \leqslant d$ the dielectric constant is a random function of position.

$$\epsilon(x,y,z) = \bar{\epsilon}[1 + \delta\epsilon(x,y,z)] \tag{1}$$

Within the slab the wave is propagated according to the (time-independent) wave equation

$$(\nabla^2 + k^2)u = -\delta\epsilon k^2 u \tag{2}$$

where $k = k_0[\bar{\epsilon}]^{1/2}$. The Born approximation for the solution to this equation is a series obtained by iteration. The first Born approximation is obtained by replacing the wave function u on the right side on Eq. (2) by $u_0 = \exp(ikz)$. The solution of Eq. (2) for the wave function at the point of observation (x,y,z) is then

$$u(x,y,z) = e^{ikz} + \frac{k^2}{4\pi}\iiint dx'dy'dz'\delta\epsilon(x',y',z') \times \frac{\exp ik\{z' + [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}\}}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}} \tag{3}$$

The integration is to be carried out over the slab in which $\delta\epsilon \neq 0$. We point out in passing that, because we have taken the mean $\bar{\epsilon}$ to be the same inside and outside the slab, no boundary reflections need be taken into account. In any case, such effects would not affect the result for the scattered field in the first Born approximation. The validity of the first Born approximation was investigated by Mintzer[7] who showed by direct comparison with the term obtained in the second iteration that Eq. (3) is valid when $k^2\langle(\delta\epsilon)^2\rangle ld \ll 1$. l is the scale of turbulence. For optical wavelengths $k_0 \sim 10^7\text{m}^{-1}$. $\langle(\delta\epsilon)^2\rangle$ may be estimated from the Gladstone-Dale relation for air $(\delta\epsilon)^2 \sim 3.6 \times 10^{-7} \langle(\delta\rho)^2\rangle/\rho_0{}^2$, where $\langle(\delta\rho)^2\rangle$ is the rms density fluctuation and ρ_0 is sea level air density. It is readily verified that the condition imposed earlier is easily satisfied in all but extreme cases. In typical experiments $l \sim 10^{-2}$m, $d \sim 10^{-1}$m and the condition reduces to $\delta\rho/\rho_0 < 0.02$.

Continuing, we express $u(x,y,z)$ in terms of u_0, modified by phase and amplitude fluctuations

$$u(x,y,z) = u_0[1 + \delta A]e^{i\delta\varphi} \approx e^{ikz}[1 + \Delta] \qquad \Delta = \delta A + i\delta\varphi \tag{4}$$

Whence

$$\Delta(x,y,z) = \frac{k^2}{4\pi}\iiint dx'dy'dz'\delta\epsilon(x',y',z') \times \frac{\exp ik\{z' - z + [(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}\}}{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}.} \tag{5}$$

Because we are treating an optical field, the scattered field at (x,y,z) is a "forward scattered" field and the integral is

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determined by the integrand within a narrow cone along the z axis with vertex at (x,y,z). Thus, we may assume $(x-x')^2+(y-y')^2 \ll (z-z')^2$ and replace the square root term in the denominator by $(z-z')$. The term in the argument of the exponential is approximately

$$[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2} \approx (z-z')\left\{1+\left(\frac{1}{2}\right)\frac{(x-x')^2+(y-y')^2}{(z-z')^2}\right\} \quad (6)$$

Whence

$$\Delta(x,y,z) = \frac{k^2}{4\pi}\iiint dx'dy'dz'\delta\epsilon(x'y'z') \times \frac{\exp ik\{(x-x')^2+(y-y')^2\}/2(z-z')}{z-z'} \quad (7)$$

The approximation for the argument of the exponential used to obtain Eq. (7) will be valid provided

$$k_0(z-z')\{[(x-x')^2+(y-y')^2]/(z-z')^2\}^2 \ll \pi \quad (8)$$

which is the condition that the higher order terms in the series expansion used in approximation of Eq. (6) will not contribute appreciably. Equation (8) may be written in terms of the vertex halfangle for the scattering cone; viz.,

$$k_0(z-z')\theta^4 \ll \pi \quad (9)$$

The first Born approximation is equivalent to the assumption of a single-scatter process. The width of the angular spectrum for single-scattered power is $\sim\lambda/l$. Thus, the approximation is subject to the weak restriction of the sagittal approximation given by

$$(z-z') \ll l^4/\lambda^3 \quad (10)$$

We note in passing that Eq. (7) is essentially the same as the expression derived by Bremmer[8] in the case $z = d$. The present procedure avoids a series of difficult transformations employed by Bremmer to obtain this result and more carefully defines the limits of validity of the result.

We now transform to cylindrical coordinates (ρ_1,θ_1,z_1) centered at $(x,y,0)$ and obtain

$$\Delta(x,y,z) = \frac{k^2}{4\pi}\int_0^d\int_0^\infty\int_0^{2\pi}\rho_1 dz_1 d\rho_1 d\varphi_1 \times \delta\epsilon(x+\rho_1\cos\varphi_1, y+\rho_1\sin\varphi_1 z_1)\frac{\exp ik\{\rho_1^2/2(z-z_1)\}}{z-z_1} \quad (11)$$

Defining

$$\delta\epsilon_\varphi(x,y,z,\rho_1,z_1) = \int_0^{2\pi}\delta\epsilon d\varphi_1 \quad (12)$$

we may express Eq. (11) in the form

$$\Delta(x,y,z) = \frac{k^2}{4\pi}\int_0^d dz_1\int_0^\infty \delta\epsilon_\varphi \times d[(ik_0)^{-1}\exp\{ik[\rho_1^2/2(z-z_1)]\}]. \quad (13)$$

The result may now be expressed as a series in k^{-1}, obtained by repeated partial integrations. If we define operator

$$\mathcal{R} = -(1/\rho_1)\partial/\partial\rho_1 \quad (14)$$

we obtain, assuming that $\delta\epsilon_\varphi$ is a physically well-behaved function, which has zero derivatives of all orders at infinity

$$\Delta(x,y,z) = \frac{ik}{4\pi}\int_0^d dz_1\sum_{n=0}^\infty\left(\frac{z-z_1}{ik}\mathcal{R}\right)^n\delta\epsilon_\varphi\bigg|_{\rho_1=0} \quad (15)$$

The leading terms in the series provide the optical phase and amplitude fluctuations. Thus,

$$\delta\varphi(x,y,z) = \frac{k}{4\pi}\int_0^d dz_1\delta\epsilon_\varphi\bigg|_{\rho=0} = \frac{k}{2}\int_0^d dz_1\delta\epsilon(x,y,z_1) \quad (16)$$

as usual, and

$$\delta A(x,y,z) = \frac{-1}{4\pi}\int_0^d dz_1(z-z_1)\frac{1}{\rho_1}\frac{\partial}{\partial\rho_1}\delta\epsilon_\varphi\bigg|_{\rho_1=0} \quad (17)$$

The region of applicability of the first terms in an equivalent series representation of Eq. (15) has been investigated in the case $d = z_1$ by Bremmer[8] who showed that for a gaussian turbulence spectrum $d\lambda/l^2 \ll 1$ is required. This condition is met by shadowgraph geometries.

It is clear that Eq. (17) exists only if the Taylor expansion of $\delta\epsilon_\varphi$ in ρ_1 about $\rho_1 = 0$ does not contain a term of first order in ρ. (This is readily seen to an expected property of $\delta\epsilon_\varphi$). Thus we obtain directly from the expansion (remembering that $\delta\epsilon_\varphi$ is not a function of ρ)

$$(1/\rho_1)(\partial/\partial\rho_1)\delta\epsilon_\varphi|_{\rho_1=0} = \tfrac{1}{2}\nabla_T^2\delta\epsilon_\varphi|_{\rho_1=0} = \pi\nabla_T^2\delta\epsilon|_{\rho_1=0} \quad (18)$$

where ∇_T^2 is the transverse Laplacian. We now obtain the fluctuation of irradiance, $I(x,y,z)$ from Eqs. (4) and (17)

$$\delta I(x,y,z) = \delta(uu^*) = 2\delta A = -\frac{1}{2}\int_0^d dz_1(z-z_1)\nabla_T^2\delta\epsilon\bigg|_{\rho_1=0} \quad (19)$$

Two special cases of interest may be obtained immediately. For $z \gg d$ we obtain

$$\delta I(x,y,z) = -\frac{z}{2}\int_0^d \nabla_T^2\delta\epsilon\bigg|_{\rho_1=0} dz_1 \quad (20)$$

This is the response function derived by Weyl and employed by the succeeding authors. For $z = d$, we obtain

$$\delta I(x,y,z) = -\frac{1}{2}\int_0^d dz_1(d-z_1)\nabla_T^2\delta\epsilon\bigg|_{\rho_1=0} \quad (21)$$

This result is identical to the result of ray optics[9,10] for $\delta I \ll 1$. (In fact, since the equations of ray optics are only valid if caustics are not considered, the ray optical equations may be regarded as valid only if $\delta I \ll 1$.)

We now return to Eq. (19) in order to obtain results for the more general case with z not much greater than d. The autocorrelation of irradiance fluctuations is

$$\langle\delta I(x',y',z')\delta I(x,y,z)\rangle = \frac{1}{4}\int_0^d\int_0^d(z-z_1)(z'-z_2)dz_1dz_2 \times \nabla_{T_1}^2\nabla_{T_2}^2 C(R)\bigg|_{\rho_1=\rho_2=0} \quad (22)$$

where for isotropic homogeneous turbulence

$$\langle\delta\epsilon(x+\rho_1\cos\varphi_1, y+\rho_1\sin\varphi_1, z_1)\delta\epsilon \times (x'+\rho_2\cos\varphi_2, y'+\rho_2\sin\varphi_2, z_2)\rangle = \langle(\delta\epsilon)^2\rangle C(R) \quad (23)$$

$$R^2 = [(x-x')+\rho_1\cos\varphi_1-\rho_2\cos\varphi_2]^2 + [(y-y')+\rho_1\sin\varphi_1-\rho_2\sin\varphi_2]^2+[z_1-z_2]^2 \quad (24)$$

Here $C(R)$ is the normalized autocorrelation of fluctuations of dielectric permittivity. It is most convenient to proceed regarding C as a function of R^2. We find

$$\nabla_{T_1}^2\nabla_{T_2}^2 C(R)\bigg|_{\rho_1=\rho_2=0} = 16\frac{\partial^2 C(R^2)}{\partial(R^2)^2}\bigg|_{\rho_1=\rho_2=0} \quad (25)$$

A straightforward manipulation now yields

$$\nabla_{T_1}^2\nabla_{T_2}^2 C(R)\bigg|_{\rho_1=\rho_2=0} = 4\left[\frac{1}{R^2}\frac{\partial^2 C}{\partial R^2}-\frac{1}{R^3}\frac{\partial C}{\partial R}\right]_{\rho_1=\rho_2=0} \quad (26)$$

and we obtain from Eq. (23)

$$\langle\delta I(x',y',z')\delta I(x,y,z)\rangle = \langle(\delta\epsilon)^2\rangle\int_0^d\int_0^d(z-z_1)(z'-z_2)\times\left[\frac{1}{R^2}\frac{\partial^2 C}{\partial R^2}-\frac{1}{R^3}\frac{\partial C}{\partial R}\right]_{\rho_1=\rho_2=0}dz_1dz_2 \quad (27)$$

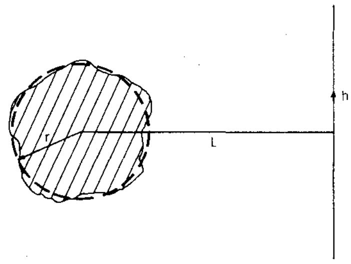

Fig. 1 Geometrical parameters for cylindrical wake.

The expression in square brackets in Eq. (27) is an even function of $z_1 - z_2 = s$ which we write as $E(s)$. Thus changing variables

$$\langle \delta I(x',y',z')\delta I(x,y,z)\rangle = \langle(\delta\epsilon)^2\rangle \int_0^d (z' - z_2)(z - z_2)dz_2 \times \int_{-z_2}^{d-z_2} E(s)ds - \langle(\delta\epsilon)^2\rangle \int_0^d (z' - z_2)dz_2 \int_{-z_2}^{d-z_2} sE(s)ds \quad (28)$$

Partial integration now reduces the double integrals. We obtain

$$\langle \delta I(x',y',z')\delta I(x,y,z)\rangle = \langle(\delta\epsilon)^2\rangle \int_0^d d\gamma E(\gamma) \times \left\{2(d-\gamma)zz' - (d^2-\gamma^2)\frac{z+z'}{2} + \frac{d^3-\gamma^3}{3} - (d-\gamma)^2\cdot\frac{z+z'}{2} + \frac{(d-\gamma)^3}{3}\right\} - \langle(\delta\epsilon)^2\rangle \int_0^d \gamma d\gamma E(\gamma)\{d-\gamma\} \quad (29)$$

However, in all cases of interest, the scale of turbulence is small compared to d, so that the range of appreciable $E(\gamma)$ is small compared to d. Whence

$$\langle \delta I(x',y',z')\delta I(x,y,z)\rangle = 2\langle(\delta\epsilon)^2\rangle \left\{zz'd - \frac{d^2(z+z')}{2} + \frac{d^3}{3}\right\} \int_0^\infty d\gamma E(\gamma) \quad (30)$$

We are usually interested in the transverse correlation. Thus, with $z = z' = D$ we obtain

$$\langle \delta I(x+\alpha,y+\beta,D)\delta I(x,y,D)\rangle = 2\langle(\delta\epsilon)^2\rangle \left\{\frac{d^3}{3} + dD^2\left[\frac{D-d}{D}\right]\right\} \int_0^\infty d\gamma \times \left\{\frac{1}{\alpha^2+\beta^2+\gamma^2} C''([\alpha^2+\beta^2+\gamma^2]^{1/2}) - \frac{1}{(\alpha^2+\beta^2+\gamma^2)^{3/2}} C'([\alpha^2+\beta^2+\gamma^2]^{1/2})\right\} \quad (31)$$

For $D = d$ we obtain a result identical to that obtained by Keller[6] and Beckmann,[9] using ray optics.

$$\langle \delta I(x+\alpha,y+\beta,D)\delta I(x,y,D)\rangle = \frac{2}{3}\langle(\delta\epsilon)^2\rangle d^3 \mathcal{J}(\alpha,\beta) \quad (32)$$

where

$$\mathcal{J}(\alpha,\beta) = \int_0^\infty d\gamma \left\{\frac{1}{(\alpha^2+\beta^2+\gamma^2)} C''([\alpha^2+\beta^2+\gamma^2]^{1/2}) - \frac{1}{(\alpha^2+\beta^2+\gamma^2)^{3/2}} C'([\alpha^2+\beta^2+\gamma^2]^{1/2})\right\} \quad (33)$$

with the trivial difference that we have the autocorrelation of irradiance (rather than amplitude fluctuations) and express the result in terms of the variation of permittivity rather than refractivity. (A similar result appears in Ref. 8, but the numerical factor is erroneous.)

For $D \gg d$, we obtain

$$\langle \delta I(x+\alpha,y+\beta,D)\delta I(x,y,D)\rangle = 2\langle(\delta\epsilon)^2\rangle \mathcal{J}(\alpha,\beta)dD^2 \quad (34)$$

which is equivalent to the formula usually employed.[1-4] It is important to note that the result expressed by Eq. (31) does not modify the usual formula in respect to the scale of irradiance autocorrelation, which depends upon (α,β); it modifies only the magnitude.

III. Application to Wake Geometry

For the cylindrical geometry illustrated in Fig. 1, it follows from Eq. (31) that the various of irradiance varies in the transverse direction in proportion to the function

$$T(h/r,r/L) = [1 - (h/r)^2]^{1/2}\{1 + \tfrac{1}{3}(r/L)(1 - (h/r)^2)\} \quad (35)$$

In Fig. 2 we plot this function. It may be observed that Fig. 2 implies that for $r/L = 0$ the irradiance fluctuation is 50% of its $h = 0$ value at $h = 0.87r$, while for $r/L = 1$ the 50% point occurs at about $h = 0.80r$. The reader should note that Eq. (35) is independent of the form of the autocorrelation of fluctuations of permittivity, $C(R)$. It should also be noted that this result does not include the effects of the laminar superlayer on the optical response; in high-density cases it is possible that this phenomenon may play a critical role.

Appendix

It has long been known on the basis of qualitative considerations[4] that shadowgraph or schlieren photographs of the turbulent wakes of ballistic models must be taken with an optical arrangement in which the photographic plate is placed closer than the focal points created by the stochastic variation of index of refraction. This is illustrated in Fig. 3. Although the situation is clear on the qualitative level, no quantitative estimate has been made of the critical distance to the focal point. The purpose of this Note is to provide a qualitative criterion, based upon the statistical properties of the medium. To this end, we consider a slab of turbulent dielectric of thickness d with index of refraction $n = 1 + \mu(\bar{r})$. μ is stochastic variable with autocorrelation $\langle\mu^2\rangle C(\bar{r})$, which we shall assume is isotropic. In the approximation of geometric optics, the incremental phase of a ray after passing through the turbulent region is

$$\int_0^d \mu dz \quad (A1)$$

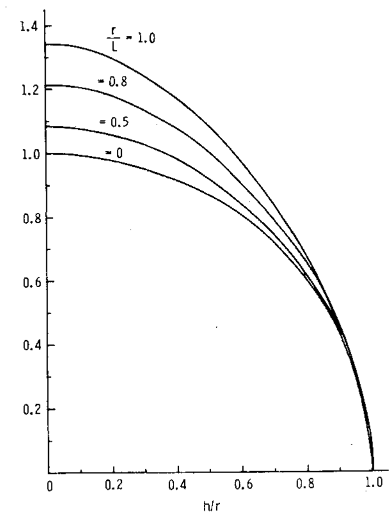

Fig. 2 The function T (h/r, r/L).

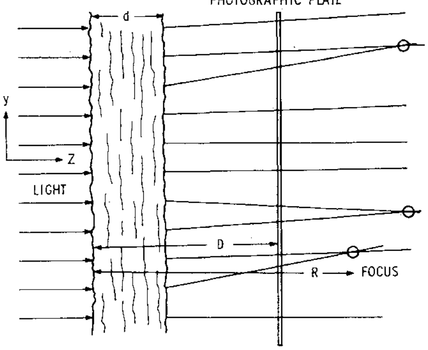

Fig. 3 Relation of photographic plate to statistical focusing in shadowgraph of turbulent wakes.

where z is taken perpendicular to the slab. Then the angular deviation of the ray in the y-z plane is

$$\theta = \int_0^d \frac{\partial \mu}{\partial y}\, dz \tag{A2}$$

In order to compute the distance R to the first focus, we observe that within the accuracy of the expansion

$$\theta(y_2) - \theta(y_1) = (\partial\theta/\partial y)(y_2 - y_1) \tag{A3}$$

we obtain from Fig. 4 with R, $D \gg d$

$$R = 1/(\partial\theta/\partial y) \tag{A4}$$

Thus, the criterion required for the optical system is

$$D \ll [(\partial\theta/\partial y)_{rms}]^{-1} \tag{A5}$$

Using standard techniques[9] we readily calculate from Eq. (A2)

$$\left\langle \left(\frac{\partial\theta}{\partial y}\right)^2 \right\rangle = \int_0^d \frac{\partial^2\mu(x_1,y_1,z_1)}{\partial y_1{}^2}\, dz_1 \int_0^d \frac{\partial^2\mu(x_2,y_2,z_2)}{\partial y_2{}^2}\, dz_2 = 2d\langle\mu^2\rangle \int_0^\infty \left.\frac{\partial^4 C}{\partial y^4}\right|_{y=x=0} dz \tag{A6}$$

If we employ a mathematical Gaussian model with $C(r) = \exp(-r^2/l^2)$, we readily obtain from Eqs. (A5) and (A6) the requirement

$$D \ll \{[1/8(\pi)^{1/2}]l^3/d\langle\mu^2\rangle\}^{1/2} \tag{A7}$$

A slightly more difficult computation is required if we employ a Kolmogoroff model

$$C(r) = \begin{cases} 1 - \frac{1}{2}l_0{}^{-2/3}l_i{}^{-4/3}r^2 & 0 < r < l_i \\ 1 - \frac{1}{2}l_0{}^{-2/3}r^{3/2} & l_i < r < 2^{3/2}l_0 \\ 0 & r > 2^{3/2}l_0 \end{cases} \tag{A8}$$

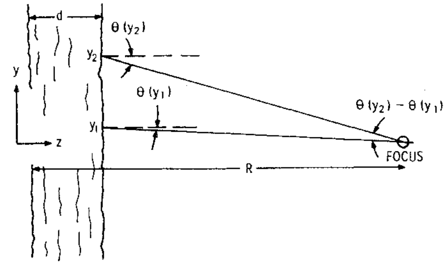

Fig. 4 Focusing geometry.

where $l_i \ll l_0$ are the "inner" and "outer" scales of turbulence, respectively. We find

$$D \ll \{7l_0{}^{2/3}l_i{}^{7/3}/8d\langle\mu^2\rangle\}^{1/2} \tag{A9}$$

The following values are taken as "typical" for hypersonic ballistics ranges: $l_0 \sim 5$ cm, $l_i \sim 0.2$ cm, $d \sim 10$ cm, $\langle\mu^2\rangle \sim 10^{-11}$. With these parameters we find $D \ll 250$ m, a condition that is "usually" met in this type of experiment.

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